

Double scaling limits and twisted non-critical superstrings

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ABSTRACT: We consider double-scaling limits of multicut solutions of certain one matrix models that are related to Calabi-Yau singularities of type A and the respective topological B model via the Dijkgraaf-Vafa correspondence. These double-scaling limits naturally lead to a bosonic string with $c \leq 1$. We argue that this non-critical string is given by the topologically twisted non-critical superstring background which provides the dual description of the double-scaled little string theory at the Calabi-Yau singularity. The algorithms developed recently to solve a generic multicut matrix model by means of the loop equations allow to show that the scaling of the higher genus terms in the matrix model free energy matches the expected behaviour in the topological B-model. This result applies to a generic matrix model singularity and the relative double-scaling limit. We use these techniques to explicitly evaluate the free energy at genus one and genus two.

KEYWORDS: Matrix Models, Topological Strings.

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1. Introduction

In [1], the large N limit of a class of $\mathcal{N} = 1$ supersymmetric $U(N)$ gauge theories was studied. The theories are in a partially confining phase where an abelian subgroup \hat{G} of the gauge group remains unconfined. The large N spectrum contains the usual weakly interacting glueballs, which are neutral under \hat{G} , and also baryonic states which are electrically and magnetically charged with respect to \hat{G} , whose mass grows like N . The models studied include the β -deformation of $\mathcal{N} = 4$ Super Yang-Mills and $\mathcal{N} = 1$ SYM coupled to a single adjoint chiral superfield with a polynomial superpotential. At some isolated points in the parameter/moduli space of the models, these baryons can become massless, and this causes the $1/N$ expansion to break down. However, it is possible to define a double-scaling limit in which N goes to infinity and the mass M_B of these states is kept constant. The crucial feature of this double-scaling limit is that there is a sector of the Hilbert space of the theory which decouples from the rest and has finite interactions which are weighted by the double-scaling parameter $1/N_{\text{eff}} \sim \sqrt{T}/M_B$, where T is the tension of the confining string. Furthermore, it was proposed in [1] that the dynamics of this emergent sector has a dual description given in terms of a non-critical superstring of the type introduced in [2]. This dual formulation has the virtue that the worldsheet theory is exactly solvable and that the background is free from Ramond-Ramond fluxes.

The exact vacuum structure and F-terms of the $\mathcal{N} = 1$ models with an adjoint chiral field and a polynomial superpotential can be analyzed by means of the Dijkgraaf-Vafa matrix model correspondence [3–5]. Indeed the proposal of [1] is based on a careful analysis of these F-terms. The breakdown of the $1/N$ expansion corresponds to a singularity of the matrix model spectral curve and therefore of the dual Calabi-Yau. The baryon states that become massless correspond to $D3$ -branes wrapping shrinking 3-cycles in the Calabi-Yau.

In [6], this analysis was extended to a more general class of singularities. Again, it was found that at these particular points in the moduli space certain states become massless and that in a suitable double-scaling limit, where the mass of these states is kept fixed, a particular sector of the theory emerges with interactions governed by the double-scaling parameter. There are two novel features in these models. First of all, contrary to the cases considered in [1], there is no supersymmetry enhancement in the double-scaling limit. This is signalled by the fact that the glueball superpotential does not vanish in the interacting sector. In fact, this is also one of the reasons why the dual string background is not determined explicitly. Secondly, in [6], some or all of the states that become massless are neutral under the abelian subgroup \hat{G} of the $U(N)$ theory which remains unconfined. As a consequence, the presence of these extra massless states may not affect the coupling constants of \hat{G} but is captured by the higher genus terms of the matrix model free energy as in [7]. These terms control certain F-term interactions of the glueball fields with the graviphoton and gravitational backgrounds [8, 9].

Another important feature that emerges from the analysis of [6] is that these large N double-scaling limits correspond to double-scaling limits of the Dijkgraaf-Vafa matrix model of the same kind that was considered in the "old matrix model" era to study $c \leq 1$ systems coupled to two-dimensional gravity [10]. In particular, in [6] it was shown that the double-scaling limits have a well-defined genus expansion in the sense that the genus g free energy of the matrix model F_g scales like $\Delta^{2-2g} \sim M_B^{2-2g}$ [6]. On the other hand, the singularities and double-scaling limits considered in [1, 6] generally fall into different universality classes from the ones usually considered in the old matrix model. It is natural to ask what is the bosonic non-critical string that corresponds to these matrix model double-scaling limits and what is the relation between the bosonic non-critical string and the non-critical superstrings that enter in the dual description of the models considered in [1]. The answer to the first question is provided by the Dijkgraaf-Vafa correspondence [3, 11, 12] that states that a generic one matrix model with superpotential $W(\Phi)$ is mapped to the topological B model on a non-compact Calabi-Yau of the form

$$uv + y^2 + W'(x)^2 + \text{deformations} = 0 \tag{1.1}$$

In taking a double-scaling limit, we tune the parameters of the superpotential and the deformation polynomial so that we are in the neighbourhood of a particular singularity of the above family of Calabi-Yaus. For instance in [1] we are close to an A_{n-1} singularity

$$uv + y^2 = x^n - \mu , \tag{1.2}$$

whereas for the $(2, 2p + 1)$ bosonic minimal model coupled to 2d gravity we would have

$$uv + y^2 + x(x - \epsilon_1)^2 \dots (x - \epsilon_p)^2 = 0 . \tag{1.3}$$

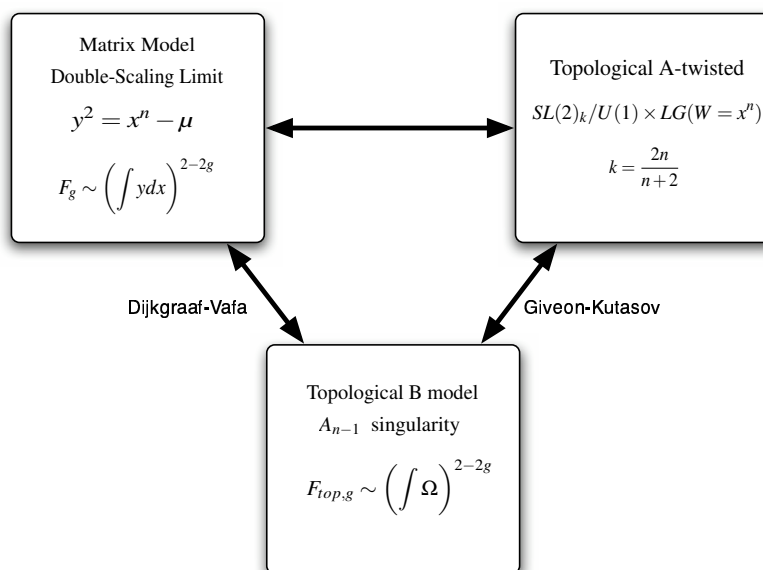


Figure 1: The bosonic non-critical string defined by the matrix model double-scaling limit at an A_{n-1} singularity corresponds to the A -twist of the above $SL(2)/U(1) \times LG$ worldsheet theory.

Therefore, we conclude that the bosonic non-critical string corresponding to the matrix model double-scaling limit of [1] is the topological B model at an A_{n-1} singularity. The case $n = 2$ corresponds to the conifold singularity.

A check that this is consistent is provided by the fact that the scaling of the matrix model free energy $F_g \sim \Delta^{2-2g}$ matches exactly the scaling of the topological B model free energy

$$F_{top,g} \sim \left(\int \Omega \right)^{2-2g} \quad g > 1 \tag{1.4}$$

where Ω is the holomorphic 3-form on the Calabi-Yau [9]. In fact, the double-scaling parameter Δ corresponds precisely to the holomorphic volume of the 3-cycles that vanish at the singularity. This is in turn proportional to M_B , the mass of the baryonic states, which come from $D3$ -branes wrapping the shrinking supersymmetric 3-cycles. Furthermore, the fact that in the double-scaling limit $F_g \sim \Delta^{2-2g}$ is a general result, it does not depend on the particular class of singularities one considers. It was derived in [6] by means of the recent algorithms to solve matrix models based on loop equations [13, 14]. These allow to consider more general classes of singularities than were previously accessible via "old matrix model" techniques.

The non-critical superstring backgrounds that appear as dual to the large N double-scaling limits studied in [1] are of the form

$$\mathbb{R}^{3,1} \times (SL(2)_k/U(1) \times LG(W = X^n)) / \mathbb{Z}, \quad k = \frac{2n}{n+2}, \tag{1.5}$$

where $LG(W)$ denotes a Landau-Ginzburg theory with superpotential W . They were initially introduced in [15] as holographic duals to the 4d double-scaled Little String Theory

(DSLST) at a CY singularity of type A_{n-1} , generalizing the proposal of [16] and previous work [17, 18]. The non-trivial part of the above background has central charge

$$\hat{c} = \hat{c}_{\text{sl}} + \hat{c}_{\text{LG}} = \frac{k+2}{k} + \frac{n-2}{n} = 3, \tag{1.6}$$

and it corresponds to the geometry [19]

$$\mu z^{-k} + uv + y^2 + x^n = 0, \tag{1.7}$$

where z, u, v, y, x are homogeneous coordinates. This is equivalent to (1.2).

We argued, using the Dijkgraaf-Vafa correspondence, that the matrix model double-scaling limits considered in [1] are equivalent to the topological B model at an A_{n-1} singularity (1.2). On the other hand, the matrix model captures the F-terms or topological terms of the 4d DSLST [1] which are given by the topological sector of the $SL(2)/U(1) \times LG$ background (1.5), [15]. Therefore, we expect the non-critical string defined by the matrix model double-scaling limits to be associated to a topologically twisted $SL(2)_k/U(1) \times LG(X^n)$ background.

This proposal fits nicely with certain known results about the topological twist of the above background in the conifold case, $n = 2$, where the LG model is trivial. In fact in [20], Ghoshal and Vafa argued that the A-twisted $N = 2$ $SL(2)/U(1)$ supersymmetric coset describes the topological B model on a deformed conifold. In [21], Mukhi and Vafa had previously shown that the above A-twisted coset at level 1 is equivalent to the $c = 1$ non-critical bosonic string compactified on a circle at self-dual radius. The open and closed sides of this map were recently analyzed in [22]. Therefore, as a direct generalization of the conifold case, we expect that the non-critical bosonic string defined by the double-scaling limit of [1] at an A_{n-1} singularity should correspond to the A-twist of the above $SL(2)/U(1) \times LG$ theory. In particular, for $n = 2$, the matrix model double-scaling limit should be equivalent to the $c = 1$ string. This fact can be checked directly on the matrix model side. Indeed, in the limit, the matrix model spectral curve becomes equivalent to that of a Gaussian model [6] which is equivalent to the $c = 1$ non-critical string [3]. This particular singularity is obtained from a 2-cut solution with a cubic superpotential in the limit where the two cuts touch each other. The fact that this singular limit should be related to the $c = 1$ non-critical string was also observed in [23].

The relation between strings on non-compact Calabi-Yaus and non-critical superstring backgrounds [19, 15] involving the $N = 2$ Kazama-Suzuki $SL(2)/U(1)$ model or its mirror, $N = 2$ Liouville theory [15, 24, 25], has been studied by several authors (see [26–28] and references therein).

Furthermore, the relation between the topological sector of six-dimensional DSLSTs defined at $K3$ singularities, the dual topologically twisted non-critical string backgrounds which generalize (1.5), and certain non-critical bosonic strings, the $(1, n)$ minimal bosonic strings has been recently studied in [29–31] (see also [32–34] for related matters).

In the paper, we are going to use the matrix model double-scaling limit to study the relative topological B model and non-critical bosonic string. In section 2, we will review the matrix models studied in [1, 6] and the respective double-scaling limits. In section 3,

we will review the proof that the genus g matrix model free energy F_g goes like Δ^{2-2g} as shown in [6]. As we said, this argument applies to a general matrix model double-scaling limit and shows that the scaling of F_g is consistent with the expected behaviour of the topological B model free energy $F_{g,top} \sim (\int \Omega)^{2-2g}$ [9]. In section 4, we will evaluate the genus one free energy F_1 at the A_{m-1} singularities considered in [1]. This gives information on the states that become massless at the singularity. In section 5, we compute the genus two free energy relevant to the matrix models considered in [1]. The result shows concretely how the double-scaling limit of F_2 depends on the details of the near-critical spectral curve. In the conifold case, the near-critical curve is a Riemann sphere and the general expression simplifies drastically and matches the well-known result.

2. The double-scaling limit

In this section, we will review the matrix model singularities and relative double-scaling limits studied in [1, 6]. Consider an $\mathcal{N} = 1$ $U(N)$ theory with a chiral adjoint field Φ and superpotential $W(\Phi)$. The classical vacua of the theory are determined by the stationary points of $W(\Phi)$

$$W(\Phi) = N \text{Tr}_N \left[\sum_{i=1}^{\ell+1} \frac{g_i}{i} \Phi^i \right]. \tag{2.1}$$

The overall factor N ensures that the superpotential scales appropriately in the 't Hooft limit. For generic values of the couplings, we find ℓ stationary points at the zeroes of

$$W'(x) = N \varepsilon \prod_{i=1}^{\ell} (x - a_i), \quad \varepsilon \equiv g_{\ell+1}. \tag{2.2}$$

The classical vacua correspond to configurations where each of the N eigenvalues of Φ takes one of the ℓ values, $\{a_i\}$, for $i = 1, \dots, \ell$. Thus vacua are related to partitions of N where $N_i \geq 0$ eigenvalues take the value a_i with $N_1 + N_2 + \dots + N_\ell = N$. Provided $N_i \geq 2$ for all i , the classical low-energy gauge group in such a vacuum is

$$\hat{G}_{\text{cl}} = \prod_{i=1}^{\ell} U(N_i) \approx \prod_{i=1}^{\ell} U(1)_i \times \text{SU}(N_i). \tag{2.3}$$

Strong-coupling dynamics will produce non-zero gluino condensates in each non-abelian factor of \hat{G}_{cl} . If we define as $W_{\alpha i}$ the chiral field strength of the $\text{SU}(N_i)$ vector multiplet in the low-energy theory, we can define a corresponding low-energy glueball superfield $S_i = -(1/32\pi^2) \langle \text{Tr}_{N_i} (W_{\alpha i} W^{\alpha i}) \rangle$ in each factor. Non-perturbative effects generate a superpotential of the form [35–37]

$$W_{\text{eff}}(S_1, \dots, S_\ell) = \sum_{j=1}^{\ell} N_j (S_j \log(\Lambda_j^3/S_j) + S_j) + 2\pi i \sum_{j=1}^{\ell} b_j S_j, \tag{2.4}$$

where the b_j are integers defined modulo N_j that label inequivalent supersymmetric vacua.

Dijkgraaf and Vafa argued that the exact superpotential of the theory can be determined by considering a matrix model with potential $W(\hat{\Phi})$ [3, 4]

$$\int d\hat{\Phi} \exp\left(-g_s^{-1} \text{Tr} W(\hat{\Phi})\right) = \exp \sum_{g=0}^{\infty} F_g g_s^{2g-2} \tag{2.5}$$

where $\hat{\Phi}$ is an $\hat{N} \times \hat{N}$ matrix in the limit $\hat{N} \rightarrow \infty$. The integral has to be understood as a saddle-point expansion around a critical point where \hat{N}_i of the eigenvalues sit in the critical point a_i . Note that \hat{N} is not related to the N from the field theory. The glueball superfields are identified with the quantities

$$S_i = g_s \hat{N}_i, \quad S = \sum_{i=1}^{\ell} S_i = g_s \hat{N} \tag{2.6}$$

in the matrix model and the exact superpotential is

$$W_{\text{eff}}(S_1, \dots, S_{\ell}) = \sum_{j=1}^{\ell} N_j \frac{\partial F_0}{\partial S_j} + 2\pi i \sum_{j=1}^{\ell} b_j S_j \tag{2.7}$$

where F_0 is the genus zero free energy of the matrix model in the planar limit.

The central object in matrix model theory is the resolvent

$$\omega(x) = \frac{1}{\hat{N}} \text{Tr} \frac{1}{x - \hat{\Phi}} \tag{2.8}$$

At leading order in the $1/\hat{N}$ expansion, $\omega(x)$ is valued on the spectral curve Σ , a hyper-elliptic Riemann surface

$$y^2 = \frac{1}{(N\varepsilon)^2} (W'(x)^2 + f_{\ell-1}(x)) \tag{2.9}$$

The numerical prefactor is chosen for convenience. In terms of this curve

$$\omega(x) = W'(x) - N\varepsilon y(x) \tag{2.10}$$

In (2.9), $f_{\ell-1}(x)$ is a polynomial of order $\ell - 1$ whose ℓ coefficients are moduli that are determined by the S_i . In general, the spectral curve can be viewed as a double-cover of the complex plane connected by ℓ cuts. For the saddle-point of interest only s of the cuts may be opened and so only s of the moduli $f_{\ell-1}(x)$ can vary. Consequently $y(x)$ has $2s$ branch points and $\ell - s$ zeros:¹

$$\Sigma : \quad y^2 = Z_m(x)^2 \sigma_{2s}(x) \tag{2.11}$$

where $\ell = m + s$ and

$$Z_m(x) = \prod_{j=1}^m (x - z_j), \quad \sigma_{2s}(x) = \prod_{j=1}^{2s} (x - \sigma_j) \tag{2.12}$$

¹Occasionally, for clarity, we indicate the order of a polynomial by a subscript.

The remaining moduli are related to the s parameters $\{S_i\}$ by (2.6)

$$S_i = g_s \hat{N}_i = N\varepsilon \oint_{A_i} y dx, \tag{2.13}$$

where the cycle A_i encircles the cut which opens out around the critical point a_i of $W(x)$.

Experience with the old matrix model teaches us that double-scaling limits can exist when the parameters in the potential are varied in such a way that combinations of branch and double points come together. In the neighbourhood of such a critical point,²

$$y^2 \longrightarrow CZ_m(x)^2 B_n(x), \tag{2.14}$$

where $z_j, b_i \rightarrow x_0$, which we can take, without loss of generality, to be $x_0 = 0$. The double-scaling limit involves first taking $a \rightarrow 0$

$$x = a\tilde{x}, \quad z_i = a\tilde{z}_i, \quad b_j = a\tilde{b}_j \tag{2.15}$$

while keeping tilded quantities fixed. In the limit, we can define the near-critical curve Σ_- :³

$$\Sigma_- : \quad y_-^2 = \tilde{Z}_m(\tilde{x})^2 \tilde{B}_n(\tilde{x}). \tag{2.16}$$

It was shown in [6], generalizing a result of [1], that in the limit $a \rightarrow 0$, in its sense as a complex manifold, the curve Σ factorizes as $\Sigma_- \cup \Sigma_+$. The complement to the near-critical curve is of the form

$$\Sigma_+ : \quad y_+^2 = x^{2m+n} F_{2s-n}(x). \tag{2.17}$$

where $F_{2s-n}(x)$ is regular at $a = 0$.

It is important to stress that the above singularities are obtained on shell [1, 6]. The family of spectral curves (2.14) corresponds to vacua of a given field theory. As such, the family satisfies the F-term equations coming from the exact superpotential (2.7) relative to the model and the choice of semiclassical vacuum. The solution to the problem of engineering these singularities on shell, namely the problem of finding a field theory model and tree level superpotential whose spectral curve exhibits the desired singularity in its moduli space, is explained in detail in [6]. The case where there are no double zeroes, $m = 0$, has been studied in [38, 39, 1]. The tree-level superpotential can be taken to be

$$W(\Phi) = N\varepsilon Tr_N [\Phi^{n+1} - \mathcal{U}\Phi], \tag{2.18}$$

and the relative on-shell spectral curve is

$$y^2 = (x^n - \mathcal{U})^2 - \mathcal{U}_c^2. \tag{2.19}$$

At each of the critical values $\mathcal{U} = \pm\mathcal{U}_c$, n branch points collide and the curve has an A_{n-1} singularity. For instance, as $\mathcal{U} \rightarrow \mathcal{U}_c$

$$y^2 \approx x^n - (\mathcal{U} - \mathcal{U}_c).$$

²We have chosen for convenience to take all the double zeros $\{z_j\}$ into the critical region.

³For polynomials, we use the notation $\tilde{f}(\tilde{x}) = \prod_i(\tilde{x} - \tilde{f}_i)$, where $f(x) = \prod_i(x - f_i)$, $x = a\tilde{x}$ and $f_i = a\tilde{f}_i$.

In the $a \rightarrow 0$ limit, it was shown in [6] that the genus g free energy gets a dominant contribution from Σ_- of the form

$$F_g \sim (Na^{(m+n/2+1)})^{2-2g} . \tag{2.20}$$

Note that in this equation N is the one from the field theory and not the matrix model \hat{N} . This motivates us to define the double-scaling limit [1, 6]

$$a \rightarrow 0, \quad N \rightarrow \infty, \quad \Delta \equiv Na^{m+n/2+1} = \text{const} . \tag{2.21}$$

Moreover, the most singular terms in a in (2.20) depend only on the near-critical curve (2.16) in a universal way.

Observe that eq. (2.20) matches the expected behaviour of the topological B model free energy at the singularity [9]. In fact, as can be seen from (2.14) and (2.15)

$$\Delta \sim N \int y dx . \tag{2.22}$$

More precisely, the double-scaling parameter is proportional to the period of the one-form $y dx$ on one of the cycles that vanish at the singularity. Moreover, this one-form corresponds to the reduction of the holomorphic 3-form Ω on the underlying Calabi-Yau geometry

$$uv + y^2 = W'(x)^2 + f(x) \tag{2.23}$$

$$\Omega = \frac{dudvdx}{\sqrt{uv - W'(x)^2 - f(x)}} . \tag{2.24}$$

This comes from the fact that 3-cycles in the Calabi-Yau correspond to two-spheres fibered over the complex x plane. In particular

$$\int \Omega \sim \int y dx \tag{2.25}$$

where Ω is integrated on a vanishing 3-cycle in the Calabi-Yau that reduces to one of the vanishing one-cycles in the matrix model spectral curve. Putting everything together, we find that

$$F_g \sim \Delta^{2-2g} \sim \left(\int y dx \right)^{2-2g} \sim \left(\int \Omega \right)^{2-2g} \tag{2.26}$$

which is precisely the behaviour we expect for the free energy of the topological B model on the Calabi-Yau [9], in agreement with the Dijkgraaf-Vafa correspondence. In section 3, we will review the proof of eq. (2.20), [6] using the algorithms of [13, 14] based on the matrix model loop equations and we will see relation (2.22) arise naturally. It is worth stressing that this result is general, in the sense that it does not depend on the specific kind of singularities one considers. In this particular respect, the methods used are more powerful than "old matrix model techniques", where one is usually limited to considering one-cut matrix model solutions.

3. The matrix model double-scaling limit: higher genus terms

In this section, we will be concerned with the behaviour of the higher genus terms of the matrix model free energy in the limit $a \rightarrow 0$. The most powerful methods for calculating the F_g involve orthogonal polynomials (see the reviews [10]); however, these techniques have only been successfully applied to the case when the near critical curve has at most two branch points (but any number of zeros). The only known way to calculate the F_g in general involves analysing the loop equations and in particular using the algorithms recently developed in [13, 14]. In the following, we will review these algorithms and the proof based on them that

$$F_g \sim \Delta^{2-2g} \tag{3.1}$$

which was given in [6]. As we said above, this is the behaviour we expect given the Dijkgraaf-Vafa correspondence between the matrix model and the topological B model.

3.1 The loop equations

The p -loop correlator, or p -point loop function, is defined as

$$\begin{aligned} W(x_1, \dots, x_p) &\equiv \hat{N}^{p-2} \left\langle \text{tr} \frac{1}{x_1 - \hat{\Phi}} \cdots \text{tr} \frac{1}{x_p - \hat{\Phi}} \right\rangle_{\text{conn}} \\ &= \frac{d}{dV}(x_p) \cdots \frac{d}{dV}(x_1) F, \quad p \geq 2, \end{aligned} \tag{3.2}$$

where we have introduced the loop insertion operator $\frac{d}{dV}(x)$, which, given a potential $V(x) = \sum_k t_k x^k$, is defined by

$$\frac{d}{dV}(x) = - \sum_k \frac{1}{x^{k+1}} \frac{d}{dt_k}. \tag{3.3}$$

The p -loop correlator has the following genus expansion

$$W(x_1, \dots, x_p) = \sum_{g=0}^{\infty} \frac{1}{\hat{N}^{2g}} W^{(g)}(x_1, \dots, x_p). \tag{3.4}$$

In [13], Eynard found a solution to the matrix model loop equations that allows to write down an expression for these multiloop correlators at any given genus in terms of a special set of Feynman diagrams. The various quantities involved depend only on the spectral curve of the matrix model and in particular one needs to evaluate residues of certain differentials at the branch points of the spectral curve.

This algorithm and its extension to calculate higher genus terms of the matrix model free energy [14] represent major progress in the solution of the matrix model via loop equations [40–43]. This is particularly important because the orthogonal polynomial approach seems to fail in the multi-cut (> 2) case. A nice feature of these algorithms is that they show directly how the information is encoded in the spectral curve. In particular, we will be able to make some precise statements on the double-scaling limits of higher genus quantities simply by studying the double-scaling limit of the spectral curve and its various differentials.

Given the matrix model spectral curve for an s -cut solution in the form (2.11) the genus zero 2-loop function is given by

$$W(x_1, x_2) = -\frac{1}{2(x_1 - x_2)^2} + \frac{\sqrt{\sigma(x_1)}}{2\sqrt{\sigma(x_2)}(x_1 - x_2)^2} - \frac{\sigma'(x_1)}{4(x_1 - x_2)\sqrt{\sigma(x_1)}\sqrt{\sigma(x_2)}} + \frac{A(x_1, x_2)}{4\sqrt{\sigma(x_1)}\sqrt{\sigma(x_2)}}, \quad (3.5)$$

where A is a symmetric polynomial given by

$$A(x_1, x_2) = \sum_{i=1}^{2s} \frac{\mathcal{L}_i(x_2)\sigma(x_1)}{x_1 - \sigma_i}, \quad (3.6)$$

where

$$\mathcal{L}_i(x_2) = \sum_{l=0}^{s-2} \mathcal{L}_{i,l}x_2^l = -\sum_{j=1}^{s-1} L_j(x_2) \int_{A_j} \frac{dx}{\sqrt{\sigma(x)}} \frac{1}{(x - \sigma_i)} \quad (3.7)$$

and s is the number of cuts. The polynomials $L_j(x)$ are related to the holomorphic 1-forms and defined in appendix A

$$\omega_j = \frac{L_j(x)dx}{\sqrt{\sigma(x)}}, \quad \int_{A_k} \omega_j = \delta_{jk}, \quad j, k = 1, \dots, s-1. \quad (3.8)$$

The genus zero 2-loop function for coincident arguments is

$$W(x_1, x_1) = \lim_{x_2 \rightarrow x_1} W(x_1, x_2) = -\frac{\sigma''(x_1)}{8\sigma(x_1)} + \frac{\sigma'(x_1)^2}{16\sigma(x_1)^2} + \frac{A(x_1, x_1)}{4\sigma(x_1)} = \sum_{i=1}^{2s} \frac{1}{16(x_1 - \sigma_i)^2} - \frac{\sigma''_i}{16\sigma'_i(x_1 - \sigma_i)} + \frac{\mathcal{L}_i(x)}{4(x_1 - \sigma_i)}, \quad (3.9)$$

where $\sigma'_i = \sigma'(\sigma_i)$, $\sigma''_i = \sigma''(\sigma_i)$. The other important object is the differential

$$dS_{2i-1}(x_1, x_2) = dS_{2i}(x_1, x_2) = \frac{\sqrt{\sigma(x_2)}}{\sqrt{\sigma(x_1)}} \left(\frac{1}{x_1 - x_2} - \frac{L_i(x_1)}{\sqrt{\sigma(x_2)}} - \sum_{j=1}^{s-1} C_j(x_2)L_j(x_1) \right) dx_1, \quad (3.10)$$

where $i = 1, \dots, s$ and

$$C_j(x_2) = \int_{A_j} \frac{dx}{\sqrt{\sigma(x)}} \frac{1}{(x - x_2)}. \quad (3.11)$$

A crucial aspect of the one-form (3.10) is that it is analytic in x_2 in the limit $x_2 \rightarrow \sigma_{2i-1}$ or σ_{2i} [13]

$$\lim_{x_2 \rightarrow \sigma_i} \frac{dS_i(x_1, x_2)}{\sqrt{\sigma(x_2)}} = \frac{1}{\sqrt{\sigma(x_1)}} \left(\frac{1}{x_1 - x_2} - \sum_{j=1}^{s-1} L_j(x_1) \int_{A_j} \frac{dx}{\sqrt{\sigma(x)}} \frac{1}{(x - x_2)} \right) dx_1. \quad (3.12)$$

The subtlety is that in the definition of (3.11), the point x_2 is taken to be outside the loop surrounding the j -th cut, whereas in (3.12), x_2 is inside the contour. Note also that

$$A(x_1, x_2) = - \sum_{i=1}^{2s} \left(\sum_{j=1}^{s-1} L_j(x_2) C_j(\sigma_i) \right) \frac{\sigma(x_1)}{x_1 - \sigma_i} \tag{3.13}$$

and in particular

$$A(x_1, \sigma_i) = \mathcal{L}_i(x_1) \sigma'(\sigma_i) . \tag{3.14}$$

The expression of $W^{(g)}(x_1, \dots, x_p)$ can be found by evaluating a series of Feynman diagrams of a cubic field theory on the spectral curve [13]. To this end, define the set $\mathcal{T}_p^{(g)}$ of all possible graphs with n external legs and with g loops. They can be described as follows: draw all *rooted* skeleton trees (trees whose vertices have valence 1, 2 or 3), with $p + 2g - 2$ edges. Draw arrows on the edges from the root towards the leaves. Then draw in all possible ways $p - 1$ external legs and g inner edges with the constraint that all the vertices of the whole graph have valence three, namely that are always three and only three edges emanating from any given vertex. Each such graph will also have some symmetry factor [13].

Each diagram is then weighted in the following way. To each arrowed edge that is part of the skeleton tree going from a vertex labelled by x_1 to a vertex labelled by x_2 associate the differential $dS(x_1, x_2)$ (3.10). To each non-arrowed edge associate a genus zero 2-loop differential $G(x_1, x_2) = W(x_1, x_2) dx_1 dx_2$ and to each *internal* vertex labelled by x_1 associate the factor $(2\varepsilon N y(x_1) dx_1)^{-1}$. For any given tree $T \in \mathcal{T}_p^{(g)}$, with root x_1 and leaves $x_j, j = 2, \dots, p$ and with $p + 2g - 2$ vertices labelled by $x'_v, v = 1, \dots, p + 2g - 2$, so that its inner edges are of the form $v_1 \rightarrow v_2$ and its outer edges are of the form $v \rightarrow j$, we define the weight of the graph as follows

$$\begin{aligned} \mathcal{W}(T) = & \frac{1}{(\varepsilon N)^{p+2g-2}} \prod_{v=1}^{p+2g-2} \sum_{i_v=1}^{2s} \text{Res}_{x'_v \rightarrow b_{i_v}} \frac{1}{2y(x'_v) dx'_v} \prod_{\text{inner edges } v \rightarrow w} dS_{i_v}(x'_v, x'_w) \\ & \times \prod_{\text{inner non-arrowed edges } v' \rightarrow w'} G_2(x'_{v'}, x'_{w'}) \prod_{\text{outer edges } v \rightarrow j} G_2(x'_v, x_j) \end{aligned} \tag{3.15}$$

In order to find an expression for $F_g, g > 1$, one should consider the same graphs relevant for $W^{(g)}(x_1)$ and do then the following [14]:

- i) Eliminate the first arrowed edge of the skeleton tree. Labelling the first vertex by x_1 and the second vertex by x_2 , this amounts to dropping the factor $dS(x_1, x_2)$.
- ii) The factor $(2\varepsilon N y(x_2) dx_2)^{-1}$ has to be dropped and replaced by

$$\frac{\int_{q_0}^{x_2} y(s) ds}{y(x_2) dx_2} . \tag{3.16}$$

Note that when evaluating the final residues at $x_2 = \sigma_i$, one needs to expand the above integral by setting $q_0 = \sigma_i$ [14]. It is also understood that the evaluation of the residues

starts from the outer branches and proceeds towards the root. This procedure does not apply for the genus one free energy whose expression has in any case been found via the loop equations in [44–46].

We will consider the $a \rightarrow 0$ limit of each element in (3.15). In this respect, it is useful to choose a new basis of 1-cycles $\{\tilde{A}_i, \tilde{B}_i\}$, $i = 1, \dots, s-1$, which is specifically adapted to the factorization $\Sigma \rightarrow \Sigma_- \cup \Sigma_+$. The subset of cycles with $i = 1, \dots, [n/2]$ vanish at the critical point while the cycles $i = [n/2] + 1, \dots, s-1$ are the remaining cycles which have zero intersection with all the vanishing cycles. Using the results in appendix A for the scaling of L , it is straightforward to argue that for a branch point b_i in the critical region

$$dS_i(x_1, x_2) \longrightarrow d\tilde{S}_i(\tilde{x}_1, \tilde{x}_2) = \frac{\sqrt{\tilde{B}(\tilde{x}_2)}}{\sqrt{\tilde{B}(\tilde{x}_1)}} \left(\frac{1}{\tilde{x}_1 - \tilde{x}_2} - \frac{\tilde{L}_i(\tilde{x}_1)}{\sqrt{\tilde{B}(\tilde{x}_2)}} - \sum_{j=1}^p \tilde{C}_j(\tilde{x}_2) \tilde{L}_j(\tilde{x}_1) \right) d\tilde{x}_1, \quad (3.17)$$

where $d\tilde{S}_i$ is the analogous differential on Σ_- , eq.(2.16), and $L_j(x) \rightarrow a^{n/2-1} \tilde{L}_j(\tilde{x})$ for $j \leq [n/2]$. Conversely, the differentials $dS_i(x_1, x_2)$ where i labels a branch point of the spectral curve that remains outside of the critical region give a vanishing contribution in the double-scaling limit. Likewise using equations (3.5), (3.6), (3.7) and (3.13) we have

$$G(x_1, x_2) = W(x_1, x_2) dx_1 dx_2 \longrightarrow \tilde{G}(\tilde{x}_1, \tilde{x}_2) = \tilde{W}(\tilde{x}_1, \tilde{x}_2) d\tilde{x}_1 d\tilde{x}_2, \quad (3.18)$$

where $\tilde{G}(\tilde{x}_1, \tilde{x}_2)$ is exactly the 2-point loop correlator on Σ_- .

So far we have seen that the double points of the near-critical curve do not play a role in taking the limit of the differentials. However, this is not the case for the final two elements of the Feynman rules

$$y dx \longrightarrow \sqrt{C} a^{m+n/2+1} y_- d\tilde{x} \quad (3.19)$$

and

$$\frac{\int_q^x y(s) ds}{y(x) dx} \longrightarrow \frac{\int_{\tilde{q}}^{\tilde{x}} y_-(\tilde{s}) d\tilde{s}}{y_-(\tilde{x}) d\tilde{x}}. \quad (3.20)$$

To summarize: what we have found is that all the relevant quantities reduce to the analogous quantities on the near-critical curve in the limit $a \rightarrow 0$. In particular, being careful with the overall scaling, the genus g free energy has the limit

$$F_g \longrightarrow C^{1-g} \Delta^{2-2g} F_g(\Sigma_-). \quad (3.21)$$

where we have emphasized that $F_g(\Sigma_-)$ depends only on Σ_- . This is the result advertized in (2.20) and the property of universality. Similarly, the genus g p -point loop functions have the limit

$$W_g(x_1, \dots, x_p) dx_1 \cdots dx_p \longrightarrow C^{1-g-p/2} \Delta^{2-2g-p} \tilde{W}_g(\tilde{x}_1, \dots, \tilde{x}_p) d\tilde{x}_1 \cdots d\tilde{x}_p. \quad (3.22)$$

4. The genus one matrix model free energy

In this section, we will consider the double-scaling limit of the genus one free energy F_1 in more detail. This term gives information on the states that become massless at the singularity [7, 19]. The genus one matrix model free energy has been studied in the context of the Dijkgraaf-Vafa correspondence in [45, 46]. In particular, the authors of [46] proposed an expression for a general multicut matrix model solution based on conformal field theory arguments by Kostov [47] and Moore [48]. This was later proved by Chekhov [44] by means of the matrix model loop equations. See also [45, 49] for an expression of the matrix model genus one free energy inspired by the correspondence with the topological B model. The general expression is given by

$$F_1 = -\frac{1}{24} \log \left(\prod_{k=1}^{2s} Z(\sigma_k) \left(\prod_{1 \leq i < j \leq 2s} (\sigma_i - \sigma_j) \right)^4 \left(\det_{i,j=1,\dots,s-1} N_{ij} \right)^{12} \right) \quad (4.1)$$

where s is the number of cuts of the matrix model solution and

$$N_{ij} = \int_{A_j} \frac{x^{i-1}}{\sqrt{\sigma(x)}} dx \quad i, j = 1, \dots, s-1 \quad (4.2)$$

are periods of the holomorphic one-forms $\frac{x^{i-1} dx}{\sqrt{\sigma(x)}}$ on the reduced spectral curve. This formula was derived in [44] by considering the genus one 1-point function $W^{(g=1)}(x)$ and explicitly inverting the relation

$$\frac{d}{dV}(x) F_1 = W^{(g=1)}(x) . \quad (4.3)$$

In the previous section, we have seen that in the double-scaling limit

$$W^{(g=1)}(x) \rightarrow \frac{1}{\Delta} \tilde{W}^{(1)}(\tilde{x}) \quad (4.4)$$

where $\tilde{W}^{(1)}(\tilde{x})$ is the genus one one-point function relative to the near-critical spectral curve Σ_-

$$y_-^2(\tilde{x}) = \tilde{Z}_m(\tilde{x}) \tilde{B}_n(\tilde{x}) . \quad (4.5)$$

We can actually absorb the factor of Δ in the definition of the curve itself

$$y_-^2 = \Delta^2 \tilde{Z}_m(\tilde{x}) \tilde{B}_n(\tilde{x}) . \quad (4.6)$$

Thus we obtain

$$\frac{d}{dV}(x) F_1 = W^{(g=1)}(x) \quad \rightarrow \quad \tilde{W}^{(1)}(\tilde{x}) = \frac{d}{d\tilde{V}}(\tilde{x}) \tilde{F}_1 . \quad (4.7)$$

We can relate the loop insertion operator $\frac{d}{dV}(x)$ to $\frac{d}{d\tilde{V}}(\tilde{x})$ as follows. Thanks to the identity [44, 14]

$$\frac{d\sigma_i}{dV}(x) = \chi_i^{(1)}(x) = \frac{1}{2N\varepsilon Z(\sigma_i)\sqrt{\sigma(x)}} \left(\frac{1}{x - \sigma_i} + \dots \right) dx \quad (4.8)$$

where the differential $\chi_i^{(1)}(x)$ is defined in (B.1), we find

$$\frac{d\sigma_i}{dV}(x) \rightarrow \frac{a}{2\Delta \tilde{Z}_m(\tilde{\sigma}_i) \sqrt{\tilde{B}_n(\tilde{x})}} \left(\frac{1}{\tilde{x} - \tilde{\sigma}_i} + \dots \right) d\tilde{x} = a \tilde{\chi}_i^{(1)}(\tilde{x}) = a \frac{d\tilde{\sigma}_i}{d\tilde{V}}(\tilde{x}) = \frac{d\sigma_i}{d\tilde{V}}(\tilde{x}).$$

This implies that

$$\frac{d}{dV}(x) \rightarrow \frac{d}{d\tilde{V}}(\tilde{x}). \quad (4.9)$$

Therefore, by (4.7), we conclude that in the double-scaling limit

$$F_1 \rightarrow \tilde{F}_1 = -\frac{1}{24} \log \left(\Delta^n \prod_{i=1}^n \tilde{Z}(\tilde{\sigma}_i) \left(\prod_{1 \leq i < j \leq n} (\tilde{\sigma}_i - \tilde{\sigma}_j) \right)^4 \left(\det_{i,j=1,\dots,[n/2]} \tilde{N}_{ij} \right)^{12} \right), \quad (4.10)$$

where \tilde{N}_{ij} are periods on the near-critical spectral curve Σ_- . This is strictly correct only modulo the addition of a constant, but this plays no role when one considers general correlators obtained from F_1 like $W^{(1)}(x)$ in (4.3). We also see that the double-scaled free energy depends in general on the structure of the near-critical spectral curve. In this respect, observe that the general expression of F_1 , eq.(4.1), depends on the basis of A -cycles we choose on the spectral curve. Upon a change of basis, which would correspond physically to an electric-magnetic duality transformation, F_1 changes non-trivially. The expression (4.10) contains an implicit choice of basis in which the degeneration of the original spectral curve Σ into $\Sigma_+ \cup \Sigma_-$ is made manifest [1, 6]. In particular, as in section 3, we choose a basis such that $[n/2]$ of the starting A -cycles shrink at the singularity and reduce to A -cycles on the near-critical spectral curve Σ_- .

In the case of the A_{n-1} singularities studied in [1], where the near-critical curve is

$$y_-^2 = \tilde{B}_n(\tilde{x}) \quad (4.11)$$

we find that in the limit $\Delta_{(n)} \rightarrow 0$

$$F_1 \rightarrow -\frac{n}{24} \log \Delta_{(n)}. \quad (4.12)$$

In particular, for the conifold singularity, $n = 2$, we retrieve the well-known result

$$F_1 = -\frac{1}{12} \log \Delta_{(2)}. \quad (4.13)$$

In the case of an "old matrix model" singularity, where m double zeroes collide with one branch point of the reduced spectral curve, σ_0 , and correspondingly the near-critical spectral curve is trivial, eq.(4.10) yields

$$F_1 \rightarrow -\frac{1}{24} \log \Delta \tilde{Z}(\tilde{\sigma}_0) \quad (4.14)$$

which is indeed consistent with the result given in [40].

The divergence of F_1 and equivalently of the topological B model free energy $F_{top,1}$ in the limit $\Delta \rightarrow 0$ indicates that there are states in the field theory that become massless at the singularity [7, 19]. Consider type IIB string theory compactified on a Calabi-Yau space in the proximity of a conifold singularity. Vafa argued that (4.13) is consistent with the appearance of a single massless 4d $\mathcal{N} = 2$ hypermultiplet in the low-energy theory [7]. Similarly, type IIB compactified on a Calabi-Yau in the proximity of an A_{n-1} singularity yields a 4d $\mathcal{N} = 2$ theory close to an Argyres-Douglas point where mutually non-local electric and magnetic particles become massless [50–52]. These extra massless particles also appear in the $\mathcal{N} = 1$ theories studied via the Dijkgraaf-Vafa matrix model [38, 1, 6].

Vafa also made a proposal about a general expression for F_1

$$\mathcal{F}_1 = F_1 + \bar{F}_1 = -\frac{1}{12} \sum_{\text{BPS states}} \log m_i^2 \tag{4.15}$$

where the sum is over BPS states of the $\mathcal{N} = 2$ 4d theory. These states can be electrically and magnetically charged and come from $D3$ -branes wrapping a supersymmetric 3-cycle C_i in the Calabi-Yau. Their mass is given by

$$m_i^2 = \frac{\int_{C_i} \Omega \cdot \int_{C_i} \bar{\Omega}}{\int_{CY} \Omega \wedge \bar{\Omega}}. \tag{4.16}$$

This is a generalization of the conifold result (4.13) where $m^2 = |\Delta|^2$ and, as stressed in [7], it might not be the full answer. For A_{n-1} singularities with n odd, the genus one expression (4.12) is indeed not matching the proposal (4.15). This is probably due to the fact that the states becoming massless are mutually non-local. It would be interesting to understand better the nature of this result.

5. Evaluation of the genus 2 free energy

For one-cut matrix model solutions the method of orthogonal polynomials allows to evaluate the matrix model free energy at all genera [10]. In the case of multicut solutions, this technique is not generally available. In order to evaluate higher genus terms in the free energy, we will resort to the recently developed algorithms that provide an exact solution to the matrix model loop equations [13, 14] that we reviewed in section 3. In particular, we will find the expression for the genus two free energy in the case where the spectral curve has no double zeroes. This is relevant for the A_{n-1} singularities considered in [1].

The genus two (5.20) and the genus one (4.10) results show that the double-scaled free energy depends in a complex way on the details of the near-critical spectral curve. However, we will see that in the simplest case, namely the conifold singularity, this dependence is trivial and the expressions simplify drastically. This is due to the fact that it is possible to choose a basis of A-cycles on the original spectral curve, before the double-scaling limit, in such a way that the the near-critical curve has genus zero, it is a Riemann sphere. Thus we recover the known result [9]

$$u^2 + v^2 + y^2 + x^2 = \mu \quad \rightarrow \quad F_2 = -\frac{1}{240\mu^2}. \tag{5.1}$$

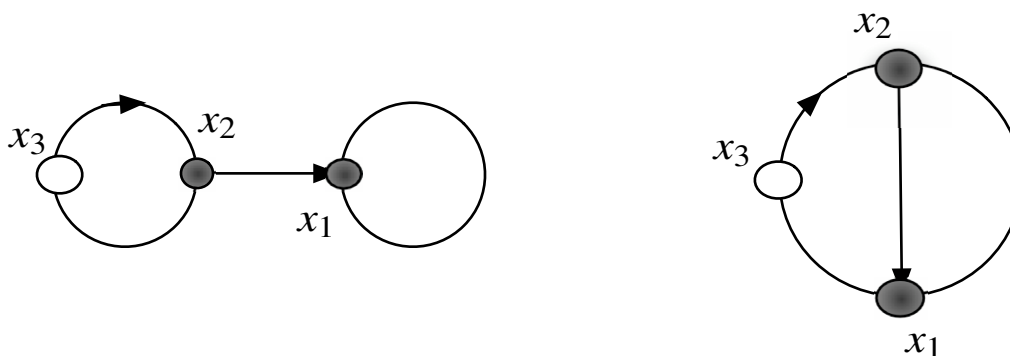


Figure 2: Diagrams (I) and (II).

The explicit evaluation of F_2 involves calculating three Feynman diagrams (see figures 2 and 3). Diagram (I) is equivalent to

$$\begin{aligned}
 (I) &= \int_{C^{x_3} > C^{x_2} > C^{x_1}} \frac{dx_3}{2\pi i} \frac{dx_2}{2\pi i} \frac{dx_1}{2\pi i} \frac{\int_{q_0}^{x_3} y(s) ds}{y(x_3) dx_3} \frac{dS(x_3, x_2)}{2y(x_2)} W(x_3, x_2) \frac{dS(x_2, x_1)}{2y(x_1)} W(x_1, x_1) \\
 &= \int_{C^{x_3} > C^{x_2}} \frac{dx_3}{2\pi i} \frac{dx_2}{2\pi i} \frac{\int_{q_0}^{x_3} y(s) ds}{y(x_3) dx_3} \frac{dS(x_3, x_2)}{2y(x_2)} W(x_3, x_2) W^{(1)}(x_2) . \quad (5.2)
 \end{aligned}$$

Diagram (II) is

$$\begin{aligned}
 (II) &= \int_{C^{x_3} > C^{x_2} > C^{x_1}} \frac{dx_3}{2\pi i} \frac{dx_2}{2\pi i} \frac{dx_1}{2\pi i} \frac{\int_{q_0}^{x_3} y(s) ds}{y(x_3) dx_3} \frac{dS(x_3, x_2)}{2y(x_2)} \frac{dS(x_2, x_1)}{2y(x_1)} W(x_3, x_1) W(x_2, x_1) \\
 &= \int_{C^{x_3} > C^{x_2}} \frac{dx_3}{2\pi i} \frac{dx_2}{2\pi i} \frac{\int_{q_0}^{x_3} y(s) ds}{y(x_3) dx_3} \frac{dS(x_3, x_2)}{2y(x_2)} W(x_3, x_2, x_2) . \quad (5.3)
 \end{aligned}$$

Similarly, diagram (III) gives

$$\begin{aligned}
 (III) &= \int_{C^{x_3}} \frac{dx_3}{2\pi i} \frac{\int_{q_0}^{x_3} y(s) ds}{y(x_3) dx_3} \int_{C^{x_1}} \frac{dx_1}{2\pi i} \frac{dS(x_3, x_1)}{2y(x_1)} W(x_1, x_1) \int_{C^{x_2}} \frac{dx_2}{2\pi i} \frac{dS(x_3, x_2)}{2y(x_2)} W(x_2, x_2) \\
 &= \int_{C^{x_3}} \frac{dx_3}{2\pi i} \frac{\int_{q_0}^{x_3} y(s) ds}{y(x_3) dx_3} W^{(1)}(x_3) W^{(1)}(x_3) . \quad (5.4)
 \end{aligned}$$

Finally, as shown in [14]

$$2F_2 = 2(I) + 2(II) + (III) . \quad (5.5)$$

In the following, we will only consider cases where the spectral curve has no double points, and we will set

$$y^2 = \varepsilon^2 \sigma_{2s}(x) . \quad (5.6)$$

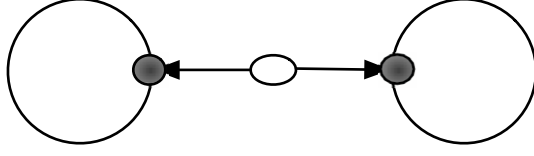


Figure 3: Diagram (III).

(I): Using the expression of $W^{(1)}(x)$ evaluated in the appendix (C.2) and the differentials $\chi_i^{(n)}(x_3)$, (B.2)(B.6), we find

$$\begin{aligned}
& \int_{C^{x_2}} \frac{dx_2}{2\pi i} \frac{dS(x_3, x_2)}{2y(x_2)} W(x_3, x_2) W^{(1)}(x_2) \\
&= \sum_{i=1}^{2s} \frac{1}{2\varepsilon} \left[\frac{1}{16\sigma'_i} G(x_3, \sigma_i) \chi_i^{(3)}(x_3) + \left(\frac{1}{16\sigma'_i} \frac{\partial}{\partial x_2} G(x_3, x_2) \Big|_{x_2=\sigma_i} - \frac{\sigma''_i}{32\sigma_i'^2} G(x_3, \sigma_i) \right) \chi_i^{(2)}(x_3) \right. \\
&\quad + \left(\frac{3\sigma_i''^2 - 2\sigma'_i \sigma_i'''}{192\sigma_i'^3} G(x_3, \sigma_i) + \frac{1}{32\sigma'_i} \frac{\partial^2}{\partial x_2^2} G(x_3, x_2) \Big|_{x_2=\sigma_i} \right. \\
&\quad \left. \left. - \frac{\sigma''_i}{32\sigma_i'^2} \frac{\partial}{\partial x_2} G(x_3, x_2) \Big|_{x_2=\sigma_i} \right) \chi_i^{(1)}(x_3) \right. \\
&\quad + G(x_3, \sigma_i) \frac{\mathcal{L}_i^{(2)}(\sigma_i)}{16\sigma'_i} \chi_i^{(1)}(x_3) + \sum_{j \neq i} \left(\frac{1}{16} \hat{\chi}_j^{(2)}(\sigma_i) + B_j \hat{\chi}_j^{(1)}(\sigma_i) \right) \frac{G(x_3, \sigma_i)}{\sigma'_i} \chi_i^{(1)}(x_3) \\
&\quad + \frac{B_i}{\sigma'_i} G(x_3, \sigma_i) \chi_i^{(2)}(x_3) \\
&\quad \left. + \left(\frac{B_i}{\sigma'_i} \frac{\partial}{\partial x_2} G(x_3, x_2) \Big|_{x_2=\sigma_i} - \frac{B_i \sigma_i''}{2\sigma_i'^2} G(x_3, \sigma_i) + B_i \frac{\mathcal{L}_i(\sigma_i)}{\sigma'_i} G(x_3, \sigma_i) \right) \chi_i^{(1)}(x_3) \right], \tag{5.7}
\end{aligned}$$

where we introduced the following notation

$$G(x_3, x_2) = \frac{\sqrt{\sigma(x_3)}}{2(x_3 - x_2)^2} - \frac{\sigma'(x_3)}{4(x_3 - x_2)\sqrt{\sigma(x_3)}} + \frac{A(x_3, x_2)}{4\sqrt{\sigma(x_3)}}, \tag{5.8}$$

$$A(x_1, x_2) = \sum_{i=1}^{2s} \frac{\mathcal{L}_i(x_2)\sigma(x_1)}{x_1 - \sigma_i}, \tag{5.9}$$

$$\hat{\chi}_k^{(n)}(x) = 2\varepsilon \sqrt{\sigma(x)} \chi_k^{(n)}(x) = \left(\frac{1}{(x - \sigma_k)^n} + \mathcal{L}_k^{(n)}(x) \right), \tag{5.10}$$

$$\mathcal{L}_i^{(2)}(x_2) = - \sum_{j=1}^{s-1} L_j(x_2) \int_{A_j} \frac{dx}{\sqrt{\sigma(x)}} \frac{1}{(x - \sigma_i)^2}, \tag{5.11}$$

$$\sigma'_i = \sigma'(\sigma_i), \quad \sigma''_i = \sigma''(\sigma_i), \quad \dots \tag{5.12}$$

We find that

$$\begin{aligned}
 (I) &= \frac{1}{4\varepsilon^2} \sum_{k=1}^{2s} \int_{C^{x_3}} \frac{dx_3}{2\pi i} \frac{\int_{A_k}^{x_3} \sqrt{\sigma(s)} ds}{\sigma(x_3)^{3/2}} \\
 &\times \sum_{i=1}^{2s} \left[\frac{1}{16\sigma'_i} \hat{G}(x_3, \sigma_i) \hat{\chi}_i^{(3)}(x_3) \right. \\
 &\quad + \left(\frac{1}{16\sigma'_i} \frac{\partial}{\partial x_2} \hat{G}(x_3, x_2) \Big|_{x_2=\sigma_i} - \frac{\sigma''_i}{32\sigma'^2_i} \hat{G}(x_3, \sigma_i) \right) \hat{\chi}_i^{(2)}(x_3) \\
 &\quad + \left(\frac{3\sigma''^2_i - 2\sigma'_i \sigma'''_i}{192\sigma'^3_i} \hat{G}(x_3, \sigma_i) + \frac{1}{32\sigma'_i} \frac{\partial^2}{\partial x_2^2} \hat{G}(x_3, x_2) \Big|_{x_2=\sigma_i} \right. \\
 &\quad \quad \left. - \frac{\sigma''_i}{32\sigma'^2_i} \frac{\partial}{\partial x_2} \hat{G}(x_3, x_2) \Big|_{x_2=\sigma_i} \right) \hat{\chi}_i^{(1)}(x_3) \\
 &\quad + \hat{G}(x_3, \sigma_i) \frac{\mathcal{L}_i^{(2)}(\sigma_i)}{16\sigma'_i} \hat{\chi}_i^{(1)}(x_3) + \sum_{j \neq i} \left(\frac{1}{16} \hat{\chi}_j^{(2)}(\sigma_i) + B_j \hat{\chi}_j^{(1)}(\sigma_i) \right) \frac{\hat{G}(x_3, \sigma_i)}{\sigma'_i} \hat{\chi}_i^{(1)}(x_3) \\
 &\quad + \frac{B_i}{\sigma'_i} \hat{G}(x_3, \sigma_i) \hat{\chi}_i^{(2)}(x_3) + \left(\frac{B_i}{\sigma'_i} \frac{\partial}{\partial x_2} \hat{G}(x_3, x_2) \Big|_{x_2=\sigma_i} - \frac{B_i \sigma''_i}{2\sigma'^2_i} \hat{G}(x_3, \sigma_i) \right. \\
 &\quad \quad \left. + B_i \frac{\mathcal{L}_i(\sigma_i)}{\sigma'_i} \hat{G}(x_3, \sigma_i) \right) \hat{\chi}_i^{(1)}(x_3) \Big] \tag{5.13}
 \end{aligned}$$

where

$$\hat{G}(x_3, \sigma_k) = \sqrt{\sigma(x_3)} G(x_3, \sigma_k) . \tag{5.14}$$

The next step is to expand

$$\frac{\int_{\sigma_k}^{x_3} \sqrt{\sigma(s)} ds}{\sigma(x_3)^{3/2}}$$

in the proximity of the branch point σ_k itself [14]. Setting $\epsilon = x - \sigma_k$, we find

$$\frac{\int_{\sigma_k}^{x_3} \sqrt{\sigma(s)} ds}{\sigma(x_3)^{3/2}} = \frac{2}{3\sigma'_k} (1 + c_{1,k}\epsilon + c_{2,k}\epsilon^2 + c_{3,k}\epsilon^3 + \mathcal{O}(\epsilon^4))$$

where

$$c_{1,k} = -\frac{3\sigma''_k}{5\sigma'_k} \quad c_{2,k} = \frac{3(8\sigma''^2_k - 5\sigma'_k \sigma'''_k)}{70\sigma'^2_k} \quad c_{3,k} = -\frac{60\sigma''^3_k - 76\sigma'_k \sigma''_k \sigma'''_k + 105\sigma'^2_k \sigma''''_k}{315\sigma'^3_k}$$

Finally

$$\begin{aligned}
 (I) = \sum_{k=1}^{2s} \frac{1}{6\varepsilon^2 \sigma'_k} & \left[\frac{1}{16\sigma'_k} \left(\frac{c_{3,k}\sigma'_k}{4} + \frac{c_{2,k}A(\sigma_k, \sigma_k)}{4} + \frac{c_{1,k}(6A^{(1,0)}(\sigma_k, \sigma_k) - \sigma_k''')}{24} \right. \right. \\
 & \left. \left. + \frac{6A^{(2,0)}(\sigma_k, \sigma_k) + 12\sigma'_k \mathcal{L}_k^{(3)}(\sigma_k) - \sigma_k''''}{48} \right) \right. \\
 & + \frac{1}{16\sigma'_k} \left(\frac{3c_{3,k}\sigma'_k + c_{2,k}\sigma_k''}{4} + \frac{c_{1,k}(6A^{(0,1)}(\sigma_k, \sigma_k) + \sigma_k''') + 18\sigma'_k \mathcal{L}_k^{(2)}(\sigma_k)}{24} \right. \\
 & + \frac{A^{(1,1)}(\sigma_k, \sigma_k)}{4} + \frac{\sigma_k'' \mathcal{L}_k^{(2)}(\sigma_k) + 3\sigma'_k \mathcal{L}_k^{(2)}(\sigma_k)'}{4} \left. \right) + \left(\frac{B_k}{\sigma'_k} - \frac{\sigma_k''}{32\sigma_k'^2} \right) \\
 & \times \left(\frac{c_{2,k}\sigma'_k}{4} + \frac{c_{1,k}A(\sigma_k, \sigma_k)}{4} + \frac{6A^{(1,0)}(\sigma_k, \sigma_k) - \sigma_k'' + 6\sigma'_k \mathcal{L}_k^{(2)}(\sigma_k)}{24} \right) \\
 & + \left(\frac{B_k}{\sigma'_k} - \frac{\sigma_k''}{32\sigma_k'^2} \right) \left(\frac{3c_{2,k}\sigma'_k}{4} + c_{1,k} \frac{\sigma_k'' + 3\sigma'_k \mathcal{L}_k(\sigma_k)}{4} \right. \\
 & \left. + \frac{6A^{(0,1)}(\sigma_k, \sigma_k) + \sigma_k'' + 6\sigma'_k \mathcal{L}_k(\sigma_k) + 18\sigma'_k \mathcal{L}'_k(\sigma_k)}{24} \right) \\
 & + \frac{1}{32\sigma'_k} \left(\frac{5c_{3,k}\sigma'_k}{2} + c_{2,k} \frac{2\sigma_k'' + 5A(\sigma_k, \sigma_k)}{2} + c_{1,k} \frac{\sigma_k'' + 4\sigma'_k \mathcal{L}_k(\sigma_k) + 10\sigma'_k \mathcal{L}'_k(\sigma_k)}{4} \right. \\
 & \left. + \frac{6A^{(0,2)}(\sigma_k, \sigma_k) + \sigma_k'''' + 6\sigma_k''' \mathcal{L}_k(\sigma_k) + 24\sigma'_k \mathcal{L}_k(\sigma_k)' + 30\sigma'_k \mathcal{L}_k''(\sigma_k)}{24} \right) \\
 & + \left(\frac{3\sigma_k''^2 - 2\sigma'_k \sigma_k'''}{192\sigma_k'^3} - \frac{B_k \sigma_k''}{2\sigma_k'^2} + B_k \frac{\mathcal{L}_k(\sigma_k)}{\sigma'_k} + \frac{\mathcal{L}_k^{(2)}(\sigma_k)}{16\sigma_k'} \right. \\
 & \left. + \sum_{j \neq k} \frac{1}{\sigma'_k} \left(\frac{1}{16} \hat{\chi}_j^{(2)}(\sigma_k) + B_j \hat{\chi}_j^{(1)}(\sigma_k) \right) \right) \left(\frac{c_{1,k}\sigma'_k}{4} + \frac{A(\sigma_k, \sigma_k)}{2} \right) \Big] \quad (5.15)
 \end{aligned}$$

where we used the notation

$$A^{(i,j)}(x, y) = \frac{\partial^i}{\partial x^i} \frac{\partial^j}{\partial y^j} A(x, y) . \quad (5.16)$$

(II): Using the expression of $W(x_2, x_2, x_3)$, eq. (C.4), we find

$$\begin{aligned}
 & \int_{C^{x_2}} \frac{dx_2}{2\pi i} \frac{dS(x_3, x_2)}{2y(x_2)} W(x_3, x_2, x_2) \\
 & = \sum_{i=1}^{2s} \left[\frac{1}{16} \chi_i^{(1)}(x_3) \chi_i^{(3)}(x_3) + \left(\frac{A(\sigma_i, \sigma_i)}{8\sigma'_i} - \frac{\sigma_i''}{32\sigma_i'} \right) \chi_i^{(1)}(x_3) \chi_i^{(2)}(x_3) \right. \\
 & + \left(\frac{A^{(1,0)}(\sigma_i, \sigma_i)}{8\sigma'_i} + \frac{A(\sigma_i, \sigma_i)^2}{16\sigma_i'^2} - \frac{\sigma'_i}{16} \sum_{j \neq i} \frac{1}{\sigma'_j(\sigma_i - \sigma_j)^2} - \frac{\sigma_i'' A(\sigma_i, \sigma_i)}{16\sigma_i'^2} \right) \chi_i^{(1)}(x_3)^2 \quad (5.17) \\
 & \left. + \sum_{j \neq i} \left(\frac{1}{16(\sigma_j - \sigma_i)^2} \left(\frac{\sigma_i'^2 + \sigma_j'^2}{\sigma'_i \sigma'_j} \right) + \frac{A(\sigma_i, \sigma_j)^2}{16\sigma'_i \sigma'_j} + \frac{A(\sigma_i, \sigma_j)(\sigma'_j - \sigma'_i)}{16(\sigma_i - \sigma_j)\sigma'_i \sigma'_j} \right) \chi_j^{(1)}(x_3) \chi_i^{(1)}(x_3) \right] .
 \end{aligned}$$

Similarly for (II) we find

$$\begin{aligned}
 (II) = \sum_{k=1}^{2s} \frac{1}{6\varepsilon^2\sigma'_k} & \left[\frac{c_{3,k}}{16} + \frac{\mathcal{L}_k^{(3)}(\sigma_k)}{16} + \frac{c_{2,k}}{16\sigma'_k} A(\sigma_k, \sigma_k) + \frac{c_{1,k}}{16\sigma'_k} A^{(1,0)}(\sigma_k, \sigma_k) \right. \\
 & + \frac{1}{32\sigma'_k} A^{(2,0)}(\sigma_k, \sigma_k) + \left(\frac{A(\sigma_k, \sigma_k)}{8\sigma'_k} - \frac{\sigma''_k}{32\sigma'_k} \right) \\
 & \times \left(c_{2,k} + \mathcal{L}_k^{(2)}(\sigma_k) + \frac{c_{1,k}A(\sigma_k, \sigma_k) + A^{(1,0)}(\sigma_k, \sigma_k)}{\sigma'_k} \right) \\
 & + \left(\frac{A^{(1,0)}(\sigma_k, \sigma_k)}{8\sigma'_k} + \frac{A(\sigma_k, \sigma_k)^2}{16\sigma'^2_k} - \frac{\sigma'_k}{16} \sum_{j \neq k} \frac{1}{\sigma'_j(\sigma_k - \sigma_j)^2} - \frac{\sigma''_k A(\sigma_k, \sigma_k)}{16\sigma'^2_k} \right) \\
 & \times \left(c_{1,k} + \frac{2A(\sigma_k, \sigma_k)}{\sigma'_k} \right) \\
 & + \sum_{j \neq k} 2 \left(\frac{1}{16(\sigma_j - \sigma_k)^2} \left(\frac{\sigma'^2_k + \sigma'^2_j}{\sigma'_k \sigma'_j} \right) + \frac{A(\sigma_k, \sigma_j)^2}{16\sigma'_k \sigma'_j} \right. \\
 & \left. + \frac{A(\sigma_k, \sigma_j)(\sigma'_j - \sigma'_k)}{16(\sigma_k - \sigma_j)\sigma'_k \sigma'_j} \right) \left(\frac{1}{\sigma_k - \sigma_j} + \frac{A(\sigma_k, \sigma_j)}{\sigma'_j} \right) \Big] \quad (5.18)
 \end{aligned}$$

(III): From eqs. (5.4) and (C.2), we find

$$\begin{aligned}
 (III) = \sum_{k=1}^{2s} \frac{1}{6\varepsilon^2\sigma'_k} & \left[\frac{c_{3,k} + 2c_{1,k}\mathcal{L}_k^{(2)}(\sigma_k) + 2\mathcal{L}_k^{(2)'}(\sigma_k)}{256} \right. \\
 & + \frac{c_{1,k}}{128} \left(\sum_{j \neq k} \frac{1}{(\sigma_k - \sigma_j)^2} + \mathcal{L}_j^{(2)}(\sigma_k) \right) + \frac{1}{128} \left(\sum_{j \neq k} -\frac{2}{(\sigma_k - \sigma_j)^3} + \mathcal{L}_j^{(2)'}(\sigma_k) \right) + \\
 & + \frac{B_k}{8} \left(c_{2,k} + \mathcal{L}_k^{(2)}(\sigma_k) + \frac{c_{1,k}A(\sigma_k, \sigma_k) + A^{(1,0)}(\sigma_k, \sigma_k)}{\sigma'_k} \right) \\
 & + \sum_{j \neq k} B_j \left(\frac{c_{1,k}}{8} \left(\frac{1}{\sigma_k - \sigma_j} + \frac{A(\sigma_j, \sigma_k)}{\sigma'_j} \right) + \frac{1}{8} \left(-\frac{1}{(\sigma_k - \sigma_j)^2} + \frac{A^{(1,0)}(\sigma_k, \sigma_j)}{\sigma'_j} \right) \right) \\
 & + \frac{B_k}{8} \sum_{j \neq k} \left(\frac{1}{(\sigma_k - \sigma_j)^2} + \mathcal{L}_j^{(2)}(\sigma_k) \right) + B_k^2 \left(c_{1,k} + 2\frac{A(\sigma_k, \sigma_k)}{\sigma'_k} \right) \\
 & \left. + 2B_k \sum_{j \neq k} B_j \left(\frac{1}{\sigma_k - \sigma_j} + \frac{A(\sigma_k, \sigma_j)}{\sigma'_j} \right) \right]. \quad (5.19)
 \end{aligned}$$

Therefore, the final expression for the genus two free energy is

$$\begin{aligned}
 F_2 = \frac{1}{\varepsilon^2} \sum_{i=1}^{2s} & \left(-\frac{157\sigma_i'^3}{15360\sigma_i'^4} + \frac{491\sigma_i''\sigma_i'''}{46080\sigma_i'^3} - \frac{35\sigma_i''''}{3072\sigma_i'^2} + \frac{35\sigma_i''^2 A(\sigma_i, \sigma_i)}{768\sigma_i'^4} - \frac{11\sigma_i'''' A(\sigma_i, \sigma_i)}{576\sigma_i'^3} - \frac{49\sigma_i'' A(\sigma_i, \sigma_i)^2}{640\sigma_i'^4} \right. \\
 & \left. + \frac{5A(\sigma_i, \sigma_i)^3}{96\sigma_i'^4} - \frac{37\sigma_i'' \mathcal{L}^{(2)}(\sigma_i)}{2560\sigma_i'^2} + \frac{A(\sigma_i, \sigma_i)\mathcal{L}_i^{(2)}(\sigma_i)}{24\sigma_i'^2} + \frac{13\mathcal{L}^{(2)'}(\sigma_i)}{1536\sigma_i'} + \frac{5\mathcal{L}_i^{(3)}(\sigma_i)}{384\sigma_i'} - \frac{11\sigma_i'' \mathcal{L}'_i(\sigma_i)}{768\sigma_i'^2} \right)
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{A(\sigma_i, \sigma_i) \mathcal{L}'_i(\sigma_i)}{32\sigma_i'^2} + \frac{5\mathcal{L}''_i(\sigma_i)}{768\sigma_i'} - \frac{47\sigma_i'' A^{(0,1)}(\sigma_i, \sigma_i)}{7680\sigma_i'^3} + \frac{5A(\sigma_i, \sigma_i) A^{(0,1)}(\sigma_i, \sigma_i)}{384\sigma_i'^3} + \frac{A^{(0,2)}(\sigma_i, \sigma_i)}{768\sigma_i'^2} \\
 & - \frac{89\sigma_i'' A^{(1,0)}(\sigma_i, \sigma_i)}{3840\sigma_i'^3} - \frac{\sigma_i'' A^{(1,0)}(\sigma_i, \sigma_i)}{160\sigma_i'^2} + \frac{7A(\sigma_i, \sigma_i) A^{(1,0)}(\sigma_i, \sigma_i)}{96\sigma_i'^3} + \frac{A^{(1,1)}(\sigma_i, \sigma_i)}{384\sigma_i'^2} + \frac{5A^{(2,0)}(\sigma_i, \sigma_i)}{768\sigma_i'^2} \\
 & + \frac{1}{\varepsilon^2} \sum_{i=1}^{2s} \sum_{j \neq i}^{2s} \left(-\frac{1}{768(\sigma_i - \sigma_j)^3 \sigma_i'} + \frac{1}{48(\sigma_i - \sigma_j)^3 \sigma_j'} + \frac{\sigma_j'}{48(\sigma_i - \sigma_j)^3 \sigma_i'^2} \right. \\
 & - \frac{\sigma_i''}{384(\sigma_i - \sigma_j)^2 \sigma_i'^2} + \frac{\sigma_i''}{10(\sigma_i - \sigma_j)^2 \sigma_i' \sigma_j'} + \frac{\sigma_j''}{1536(\sigma_i - \sigma_j)^2 \sigma_i' \sigma_j'} + \frac{\sigma_i'' \sigma_j''}{384(\sigma_i - \sigma_j) \sigma_i'^2 \sigma_j'} \\
 & + \frac{A(\sigma_i, \sigma_i)}{128(\sigma_i - \sigma_j)^2 \sigma_i'^2} - \frac{A(\sigma_i, \sigma_i)}{3(\sigma_i - \sigma_j)^2 \sigma_i' \sigma_j'} - \frac{\sigma_j'' A(\sigma_i, \sigma_i)}{128(\sigma_i - \sigma_j) \sigma_i'^2 \sigma_j'} \\
 & + \frac{A(\sigma_i, \sigma_j)}{24(\sigma_i - \sigma_j)^2 \sigma_i'^2} + \frac{A(\sigma_i, \sigma_j)}{48(\sigma_i - \sigma_j)^2 \sigma_j'^2} - \frac{A(\sigma_i, \sigma_j)}{48(\sigma_i - \sigma_j)^2 \sigma_i' \sigma_j'} + \frac{\sigma_i'' \sigma_j'' A(\sigma_i, \sigma_j)}{512\sigma_i'^2 \sigma_j'^2} + \frac{\sigma_i'' \sigma_j'' A(\sigma_i, \sigma_j)}{1536\sigma_i'^3 \sigma_j'} \\
 & - \frac{\sigma_j'' A(\sigma_i, \sigma_i) A(\sigma_i, \sigma_j)}{192\sigma_i'^2 \sigma_j'^2} - \frac{\sigma_j'' A(\sigma_i, \sigma_i) A(\sigma_i, \sigma_j)}{384\sigma_i'^3 \sigma_j'} - \frac{A(\sigma_i, \sigma_j)^2}{48(\sigma_i - \sigma_j) \sigma_i' \sigma_j'^2} + \frac{A(\sigma_i, \sigma_j)^2}{24(\sigma_i - \sigma_j) \sigma_i'^2 \sigma_j'} + \frac{A(\sigma_i, \sigma_j)^3}{48\sigma_i'^2 \sigma_j'^2} \\
 & - \frac{A(\sigma_j, \sigma_j)}{384(\sigma_i - \sigma_j)^2 \sigma_i' \sigma_j'} - \frac{\sigma_i'' A(\sigma_j, \sigma_j)}{96(\sigma_i - \sigma_j) \sigma_i'^2 \sigma_j'} + \frac{A(\sigma_i, \sigma_i) A(\sigma_j, \sigma_j)}{32(\sigma_i - \sigma_j) \sigma_i'^2 \sigma_j'} - \frac{\sigma_i'' A(\sigma_i, \sigma_j) A(\sigma_j, \sigma_j)}{128\sigma_i'^2 \sigma_j'^2} \\
 & - \frac{\sigma_i'' A(\sigma_i, \sigma_j) A(\sigma_j, \sigma_j)}{384\sigma_i'^3 \sigma_j'} + \frac{A(\sigma_i, \sigma_i) A(\sigma_i, \sigma_j) A(\sigma_j, \sigma_j)}{48\sigma_i'^2 \sigma_j'^2} + \frac{A(\sigma_i, \sigma_i) A(\sigma_i, \sigma_j) A(\sigma_j, \sigma_j)}{96\sigma_i'^3 \sigma_j'} \\
 & \left. - \frac{\sigma_i'' \mathcal{L}_j^{(2)}(\sigma_i)}{512\sigma_i'^2} + \frac{A(\sigma_i, \sigma_i) \mathcal{L}_j^{(2)}(\sigma_i)}{192\sigma_i'^2} + \frac{\mathcal{L}_j^{(2)'}(\sigma_i)}{1536\sigma_i'} - \frac{\sigma_j'' A^{(0,1)}(\sigma_i, \sigma_j)}{1536\sigma_i' \sigma_j'^2} + \frac{A(\sigma_j, \sigma_j) A^{(0,1)}(\sigma_i, \sigma_j)}{384\sigma_i' \sigma_j'^2} \right) \quad (5.20)
 \end{aligned}$$

Let us consider the limit

$$y^2 = \varepsilon^2 \sigma_{2s}(x) \longrightarrow \varepsilon^2 (x^s - a^s), \quad a \rightarrow 0 \quad (5.21)$$

where s of the branch points come together. In the double-scaling limit

$$a \rightarrow 0, \quad \varepsilon \rightarrow \infty, \quad \Delta = \varepsilon a^{s/2+1} = cnst \quad (5.22)$$

we find

$$F_2 \longrightarrow \frac{F_g(\Sigma_-)}{\Delta^2}, \quad \Sigma_- : y_-^2 = \tilde{x}^s - \tilde{a}^s \quad (5.23)$$

as explained in section 3. However, the final result will not simplify in general. In fact, it depends on the details of the near-critical spectral curve, which has genus $[(s-1)/2]$. An exception to this is given by the case of the conifold singularity, where the original spectral curve becomes in the limit

$$y^2 \approx \varepsilon^2 (x-a)(x-b), \quad a, b \rightarrow 0, \quad (5.24)$$

which is essentially the spectral curve associated to a Gaussian matrix model [6]. As in section 3, it is convenient to choose one of the A-cycles of the original spectral curve to be a loop encircling the cut going from branch point a to branch point b . This cycle will reduce to an A-cycle on the near-critical spectral curve

$$y_-^2 = \tilde{\sigma}(\tilde{x}) = (\tilde{x} - \tilde{a})(\tilde{x} - \tilde{b}) \quad (5.25)$$

where, as before, the tilded quantities are finite in the limit $a, b \rightarrow 0$. The above near-critical curve is actually a Riemann sphere. In particular, one can check by evaluating the residues of all the integrands at infinity that all periods of the form

$$\int_A \frac{d\tilde{x}}{\sqrt{\tilde{\sigma}(\tilde{x})}} \frac{1}{(\tilde{x} - \tilde{\sigma}_i)^n} \tag{5.26}$$

are zero. The expression of F_2 simplifies dramatically

$$F_2 \rightarrow -\frac{4}{15\varepsilon^2(a-b)^4} = -\frac{1}{240S^2}, \tag{5.27}$$

where

$$S = \int_A y dx \rightarrow \frac{\varepsilon(a-b)^2}{8} \sim \Delta, \tag{5.28}$$

and in terms of S the genus zero free energy is given by

$$F_0 \approx \frac{1}{2} S \frac{\partial F_0}{\partial S} \approx \frac{1}{2} S^2 \log S. \tag{5.29}$$

Thus, (5.27) indeed matches the expected result for the genus two free energy at a conifold singularity [9], which is equivalent to the $c = 1$ non-critical bosonic string [20]. This particular singularity is obtained from a 2-cut solution with a cubic superpotential in the limit where the two cuts touch each other. The fact that this limit should be equivalent to the $c = 1$ non-critical string was also observed in [23].

6. Conclusion

The class of matrix model DSLs that are associated to the large N field theory DSLs introduced in [1] define a class of $c \leq 1$ non-critical bosonic strings [6]. They fall into different universality classes from the ones usually considered in the old matrix model. We argued that these non-critical bosonic strings are related to the topologically twisted non-critical superstring backgrounds of the form $SL(2)/U(1) \times LG(X^n)$ that are dual to the large N double-scaled field theory and the associated four-dimensional double-scaled LST at the corresponding A_{n-1} singularity.

To study the matrix models, and the relevant multicut solutions, we used the techniques of Chekhov and Eynard based on loop equations. We showed in general that the scaling of the higher genus terms in the perturbative expansion of the matrix model free energy matches precisely the scaling of the topological B model free energy in the vicinity of the Calabi-Yau singularity, which is consistent with the Dijkgraaf-Vafa correspondence.

We also evaluated the genus one and two terms explicitly for the spectral curves that are related to A_{n-1} singularities, recovering the conifold result in the $n = 2$ case. These techniques allow to study multicut solutions where the "old matrix model" tools are not generally available, but further work would be needed to find the exact expression of the perturbative matrix model free energy at all orders for the A_{n-1} singularities with $n > 2$. The direct computation of higher order terms is more cumbersome but could be performed with the aid of a computer. One might be able to find an exact expression by deriving

recursion relations for the coefficients of the genus expansion for this particular class of singularities. However, this might require a different approach from the loop equations, in particular an extension of orthogonal polynomial techniques to multicut matrix model solutions. It would also be interesting to check if this perturbative series needs a non-perturbative completion like in the conifold case. Such completion should correspond to D -brane effects on the non-critical string side as in [53]. One should also perform the topological twist of the $SL(2)/U(1) \times LG$ model and determine the non-critical bosonic string explicitly.

Finally, it would certainly be interesting to use independent techniques to rederive the topological quantities evaluated in the paper and in particular find the relation between the BCOV algorithm [9] to evaluate the topological B model free energy for the non-compact Calabi-Yaus that admit a matrix model description and the algorithm of Eynard and Chekhov for the matrix model free energy. This point was recently addressed in [54]. Showing the equivalence of the two procedures would amount to a proof of the Dijkgraaf-Vafa conjecture at every order in the genus expansion.

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A. Some double-scaling formulae

In this appendix, we consider the double-scaling limit of various quantities defined on the curve Σ (2.11). This is most conveniently done in the basis $\{\tilde{A}_i, \tilde{B}_i\}$ of 1-cycles described in section 3. In particular, for $i \leq [n/2]$ these are cycles on the near-critical curve Σ_- in the double-scaling limit.

The key quantities that we will need are the periods

$$M_{ij} = \oint_{\tilde{B}_j} \frac{x^{i-1}}{\sqrt{\sigma(x)}} dx, \quad N_{ij} = \oint_{\tilde{A}_j} \frac{x^{i-1}}{\sqrt{\sigma(x)}} dx. \tag{A.1}$$

First of all, let us focus on N_{ij} where $j \leq [n/2]$, but i arbitrary. By a simple scaling argument, as $a \rightarrow 0$,

$$N_{ij} = \int_{b_{(j)}^-}^{b_{(j)}^+} \frac{x^{i-1}}{\sqrt{B(x)}} dx \longrightarrow a^{i-n/2} \int_{\tilde{b}_{(j)}^-}^{\tilde{b}_{(j)}^+} \frac{\tilde{x}^{i-1}}{\sqrt{\tilde{B}(\tilde{x})}} d\tilde{x} = a^{i-n/2} f_{ij}^{(N)}(\tilde{b}_l), \tag{A.2}$$

for some function $f_{ij}^{(N)}$ of the branch points of Σ_- . Here, $b_{(j)}^\pm$ are the two branch points enclosed by the cycle \tilde{A}_j . A similar argument shows that M_{ij} scales in the same way:

$$M_{ij} \longrightarrow a^{i-n/2} f_{ij}^{(M)}(\tilde{b}_l). \tag{A.3}$$

So both N_{ij} and M_{ij} , for $i, j, \leq [n/2]$, diverge in the limit $a \rightarrow 0$. On the contrary, by using a similar argument, it is not difficult to see that, for $j > [n/2]$, N_{ij} and M_{ij} are analytic as $a \rightarrow 0$ since the integrals are over non-vanishing cycles.

In summary, in the limit $a \rightarrow 0$, the matrices N and M will have the following block structure

$$N \longrightarrow \begin{pmatrix} N_{--} & N_{-+}^{(0)} \\ 0 & N_{++}^{(0)} \end{pmatrix}, \quad M \longrightarrow \begin{pmatrix} M_{--} & M_{-+}^{(0)} \\ 0 & M_{++}^{(0)} \end{pmatrix}, \quad (\text{A.4})$$

where by $-$ or $+$ we denote indices in the ranges $\{1, \dots, [n/2]\}$ and $\{[n/2] + 1, \dots, s - 1\}$ respectively. In (A.4), N_{--} and M_{--} are divergent while the remaining quantities are finite as $a \rightarrow 0$.

We also need the inverse $L = N^{-1}$. In the text, we use the polynomials $L_j(x) = \sum_{k=1}^{s-1} L_{jk} x^{k-1}$, which enter the expression of the holomorphic 1-forms associated to our basis of 1-cycles,

$$\oint_{\tilde{A}_i} \omega_j = \delta_{ij}. \quad (\text{A.5})$$

These 1-forms are equal to

$$\omega_j(x) = \frac{L_j(x)}{\sqrt{\sigma(x)}} dx = \frac{\sum_{k=1}^{s-1} L_{jk} x^{k-1}}{\sqrt{\sigma(x)}} dx, \quad \oint_{A_i} \omega_j(x) = \delta_{ij} \quad (\text{A.6})$$

where $i, j = 1, \dots, s - 1$. From the behaviour of N in the limit $a \rightarrow 0$, we have

$$L = N^{-1} \longrightarrow \begin{pmatrix} N_{--}^{-1} & \mathcal{N} \\ 0 & (N_{++}^{(0)})^{-1} \end{pmatrix}, \quad \mathcal{N} = -N_{--}^{-1} N_{-+}^{(0)} (N_{++}^{(0)})^{-1}. \quad (\text{A.7})$$

Since N_{--} is singular we see that L is block diagonal in the limit $a \rightarrow 0$. This is just an expression of the fact that the curve factorizes $\Sigma \rightarrow \Sigma_- \cup \Sigma_+$ as $a \rightarrow 0$. In this limit, using the scaling of elements of L_{jk} , we find, for $j \leq [n/2]$,

$$\omega_j \longrightarrow \frac{\sum_{k=1}^{[n/2]} (f^{(N)})_{jk}^{-1} \tilde{x}^{k-1}}{\sqrt{\tilde{B}(\tilde{x})}} d\tilde{x} = \tilde{\omega}_j. \quad (\text{A.8})$$

the holomorphic 1-forms of Σ_- . While for $j > [n/2]$,

$$\omega_j \longrightarrow \frac{\sum_{k>[n/2]}^{s-1} (N_{++}^{(0)})_{jk}^{-1} x^{k-n/2-1}}{\sqrt{F(x)}} dx, \quad (\text{A.9})$$

are the holomorphic 1-forms of Σ_+ .

B. The explicit expression of $\chi_i^{(n)}(p)$

Using the formalism developed in [13] and [14] to solve the matrix model loop equations, one can easily find the expression of the differentials $\chi_i^{(n)}(p)$ defined by

$$\left(\hat{K} - 2W_0(p) \right) \chi_i^{(n)}(p) = \frac{1}{(p - \sigma_i)^n}, \quad (\text{B.1})$$

where σ_i is a branch point of the matrix model spectral curve. These 1-differentials appear quite naturally in the expression of higher loop correlators in the matrix model and in the integration steps leading to F_2 . We have

$$\chi_i^{(n)}(p) = \text{Res}_{q \rightarrow \sigma_i} \left(\frac{dS_i(p, q)}{2y(q)} \frac{1}{(q - \sigma_i)^n} \right) \quad (\text{B.2})$$

Given the expression of $dS_i(p, q)$, we can easily perform a Taylor expansion in q around the branch point σ_i and find the residue. We will mainly consider the case where the matrix model spectral curve has no double points, setting

$$y^2 = \varepsilon^2 \sigma_{2s}(x) . \quad (\text{B.3})$$

In this case

$$\frac{dS_i(p, q)}{y(q)} = \frac{dS_i(p, q)}{\varepsilon \sqrt{\sigma(q)}} = \frac{1}{\varepsilon \sqrt{\sigma(p)}} \left(\frac{1}{p - q} - \sum_{j=1}^{s-1} L_j(p) \int_{A_j} \frac{dx}{\sqrt{\sigma(x)}} \frac{1}{(x - q)} \right) dp$$

Note also that the expression in brackets is analytic in q . Then, for instance, we find that

$$\begin{aligned} \chi_i^{(1)}(p) &= \frac{1}{2\varepsilon \sqrt{\sigma(p)}} \left(\frac{1}{p - \sigma_i} - \sum_{j=1}^{s-1} L_j(p) \int_{A_j} \frac{dx}{\sqrt{\sigma(x)}} \frac{1}{(x - \sigma_i)} \right) dp \\ &= \frac{1}{2\varepsilon \sqrt{\sigma(p)}} \left(\frac{1}{p - \sigma_i} + \mathcal{L}_i(p) \right) dp = \frac{1}{2\varepsilon \sqrt{\sigma(p)}} \left(\frac{1}{p - \sigma_i} + \frac{A(p, \sigma_i)}{\sigma_i'} \right) dp \end{aligned} \quad (\text{B.4})$$

$$\chi_i^{(2)}(p) = \frac{1}{2\varepsilon \sqrt{\sigma(p)}} \left(\frac{1}{(p - \sigma_i)^2} - \sum_{j=1}^{s-1} L_j(p) \int_{A_j} \frac{dx}{\sqrt{\sigma(x)}} \frac{1}{(x - \sigma_i)^2} \right) dp \quad (\text{B.5})$$

and in general

$$\chi_i^{(n)}(p) = \frac{1}{2\varepsilon \sqrt{\sigma(p)}} \frac{1}{(n-1)!} \frac{d^{n-1}}{dq^{n-1}} \left(\frac{1}{p - q} - \sum_{j=1}^{s-1} L_j(p) \int_{A_j} \frac{dx}{\sqrt{\sigma(x)}} \frac{1}{(x - q)} \right) \Big|_{q=\sigma_i} dp \quad (\text{B.6})$$

The above expressions can be generalized to the case where the spectral curve is of the form

$$y^2 = M(x)^2 \sigma(x) , \quad (\text{B.7})$$

$$\chi_i^{(n)}(p) = \frac{1}{2\sqrt{\sigma(p)}} \frac{1}{(n-1)!} \frac{d^{n-1}}{dq^{n-1}} \frac{1}{M(q)} \left(\frac{1}{p - q} - \sum_{j=1}^{s-1} L_j(p) \int_{A_j} \frac{dx}{\sqrt{\sigma(x)}} \frac{1}{(x - q)} \right) \Big|_{q=\sigma_i} dp . \quad (\text{B.8})$$

C. Evaluation of $W^{(1)}(p)$ and $W(p, p, q)$

In this section, we are going to evaluate two loop-functions whose expression is needed later for F_2 , the genus one one-loop function $W^{(1)}(p)$ and the genus zero three-loop function

$W(p, p, q)$. Let us start from $W^{(1)}(p)$. Using the diagrammatic rules of [13] we find

$$W^{(1)}(x_2) = \sum_{i=1}^{2s} \text{Res}_{x_1 \rightarrow \sigma_i} \left(\frac{dS_i(x_2, x_1)}{2y(x_1)} W(x_1, x_1) \right) \quad (\text{C.1})$$

$$\begin{aligned} &= \sum_{i=1}^{2s} \text{Res}_{x_1 \rightarrow \sigma_i} \left[\frac{dS_i(x_2, x_1)}{2y(x_1)} \left(\frac{1}{16(x_1 - \sigma_i)^2} + \frac{B_i}{x_1 - \sigma_i} \right) \right] \\ &= \sum_{i=1}^{2s} \frac{1}{16} \chi_i^{(2)}(x_2) + B_i \chi_i^{(1)}(x_2), \end{aligned} \quad (\text{C.2})$$

where

$$B_i \equiv \left(-\frac{\sigma''(\sigma_i)}{8\sigma'(\sigma_i)} + \sum_{j \neq i} \frac{1}{8(\sigma_i - \sigma_j)} + \frac{A(\sigma_i, \sigma_i)}{4\sigma'(\sigma_i)} \right) = \left(-\frac{\sigma''(\sigma_i)}{16\sigma'(\sigma_i)} + \frac{A(\sigma_i, \sigma_i)}{4\sigma'(\sigma_i)} \right). \quad (\text{C.3})$$

This is exactly the expression given for instance in [44], once we use the identity

$$A(\sigma_i, \sigma_i) = \mathcal{L}_i(\sigma_i) \sigma'(\sigma_i).$$

Then, let us evaluate the genus zero 3-loop function with two coincident arguments $W(x_2, x_2, x_3)$. We find

$$\begin{aligned} W(x_2, x_2, x_3) &= \sum_{i=1}^{2s} \text{Res}_{x_1 \rightarrow \sigma_i} \left(\frac{dS_i(x_2, x_1)}{2y(x_1)} W(x_1, x_2) W(x_1, x_3) \right) \\ &= \sum_{i=1}^{2s} \text{Res}_{x_1 \rightarrow \sigma_i} \left[\frac{dS_i(x_2, x_1)}{2y(x_1)} \left(\frac{\sigma'(\sigma_i)}{4(x_2 - \sigma_i)\sqrt{\sigma(x_2)}} + \frac{A(x_2, \sigma_i)}{4\sqrt{\sigma(x_2)}} \right) \right. \\ &\quad \left. \left(\frac{\sigma'(\sigma_i)}{4(x_3 - \sigma_i)\sqrt{\sigma(x_3)}} + \frac{A(x_3, \sigma_i)}{4\sqrt{\sigma(x_3)}} \right) \frac{1}{\sigma'(\sigma_i)(x_1 - \sigma_i)} \right] \\ &= \sum_{i=1}^{2s} \left(\frac{\sigma'(\sigma_i)}{4(x_2 - \sigma_i)\sqrt{\sigma(x_2)}} + \frac{A(x_2, \sigma_i)}{4\sqrt{\sigma(x_2)}} \right) \\ &\quad \times \left(\frac{\sigma'(\sigma_i)}{4(x_3 - \sigma_i)\sqrt{\sigma(x_3)}} + \frac{A(x_3, \sigma_i)}{4\sqrt{\sigma(x_3)}} \right) \frac{\chi_i^{(1)}(x_2)}{\sigma'(\sigma_i)} \\ &= \sum_{i=1}^{2s} \frac{\varepsilon^2 \sigma'(\sigma_i)}{4} \chi_i^{(1)}(x_2)^2 \chi_i^{(1)}(x_3). \end{aligned} \quad (\text{C.4})$$

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